Synthetic Running Coupling of QCD

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Based on a study of the analytic running coupling obtained from the standard perturbation theory results up to four-loop order, the QCD "synthetic" running coupling α_{syn} is built. In so doing the perturbative time-like discontinuity is preserved and nonperturbative contributions not only remove the nonphysical singularities of the perturbation theory in the infrared region but also decrease rapidly in the ultraviolet region. In the framework of the approach, on the one hand, the running coupling is enhanced at zero and, on the other hand, the dynamical gluon mass m_g arises. Fixing the parameter which characterize the infrared enhancement corresponding to the string tension σ and normalization, say, at M_τ completely define the synthetic running coupling. In this case the dynamical gluon mass appears to be fixed and the higher loop stabilization property of m_g is observed. For $\sigma = (0.42 \text{ GeV})^2$ and $\alpha_{\text{syn}}(M_\tau^2) = 0.33 \pm 0.01$ it is obtained that $m_g = 530 \pm 80$ MeV.

PACS numbers: 12.38.-t, 12.38.Aw, 11.15.Tk

I. INTRODUCTION

The paper is devoted to description of the running coupling (the invariant charge) of QCD in which on the basis of the perturbative study up to four-loop order the attempt has been undertaken to include the nonperturbative contributions in such a way that nonphysical singularities of the perturbation theory in the infrared region be cancelled and the essentials of the QCD dynamics for all energy scales accumulated in its framework.

Doing that it seems notably attractive, on the theoretical grounds of the analytic running coupling of QCD, to build the running coupling including additional nonperturbative terms. We will consider a variation of such couplings, the "synthetic" running coupling $\alpha_{\text{syn}}(Q^2)$, which on the practical grounds of the perturbation theory successfully used for description of the region of large Q^2 (short distances physics), contains the nonperturbative terms, determining the main features of QCD in low Q^2 region (long distances physics) without the dramatic changes of these qualitatively different regimes.

The analytic approach in QFT was formulated late in the 50s in Refs. 1 and 2 for QED and other theories. For QCD the analytic approach was applied in [3, 4]. Indicate Refs. 5, 6, 7, 8, 9, 10, 11, 12 in which the analytic approach and its applications to QCD are considering. The analyticity requirement, which follows from the general principles of QFT, enables one to resolve the difficulties connected with nonphysical singularities of the perturbation theory in the infrared region. In the "analytically improved" running coupling these singularities are cancelled out by the nonperturbative contributions. In the ultraviolet region the nonperturbative contributions rapidly decrease and the perturbative contributions are decisive. Nonetheless, the behavior of the nonperturbative contributions in the ultraviolet region originating from the procedure is of a considerable interest. Their behavior appears to be important in the construction of the synthetic running coupling.

Whereas for the one-loop case a separation of the analytic running coupling into the perturbative and the nonperturbative components and behavior of the nonperturbative component are obvious, for the multi-loop analytic running coupling this is not the case. For the two-loop case such separation in an explicit form and study of the nonperturbative component of the analytic running coupling was made in Ref. 13, for the three-loop case it was made in [14, 15], and for the four-loop case it was made in [16, 17]. The nonperturbative contributions in the analytic running coupling were extracted explicitly and their expansion in powers of Λ^2/Q^2 was obtained. The effective method of the precise calculation of the analytic running coupling was developed on a basis of this expansion.

It is known [3] that the analytic running coupling is finite at zero. In Refs. 16 and 17 it was shown that the finiteness of the analytic coupling at zero was a consequence of the asymptotic freedom property of the initial perturbation theory, $\alpha_{\text{an}}^{\text{n-loop}}(0) = 4\pi/b_0 \simeq 1.396$. The running coupling is finite at zero also in the case of "freezing" of the interaction [18]. The loss of Q^2-dependence at low Q^2 takes place for a number of different definitions of the running coupling [19]. However, with such infrared behavior of the running coupling the description of the confinement and the dynamical breaking of the chiral symmetry is not immediate. The behavior of the running coupling corresponding to the infrared enhancement of the interaction seems to be appropriate for description of these most important properties of QCD. In general the running coupling constant is ambiguously determined. In the perturbation theory it depends on the renormalization scheme choice. Besides it can depend on the nonperturbative contributions in an observable used for definition of the running coupling. In the so-called nonperturbative V-scheme for the running coupling the singular behavior $\alpha_V \sim 1/Q^2$ at $Q^2 \to 0$ corresponds to the linear confining quark-antiquark static potential at large distances with the universal string tension param-

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eter σ. Point at Ref. 20 where the behavior of the running coupling of the form $\sim 1/Q^2$ for all Q^2 takes place for a perturbative treatment of theories with permanent confinement. In QCD the static potential is defined in an explicit gauge invariant form through the vacuum expectation value of the Wilson loop. Its calculation on the lattice in quenched approximation at large distances r completely corresponds to the string picture of heavy quarks interaction [21, 22] with confining term σr and a γ/r correction with a coefficient γ as predicted by the bosonic string theory. The synthetic running coupling under consideration also belongs to the singular type couplings, $\alpha_{\mathrm{syn}} \sim 1/Q^2$ at $Q^2 \to 0$, and yet it has its own motivation related to the study of the nonperturbative contributions at $Q^2 \to \infty$. The synthetic running coupling appears practically to coincide [23] with the initial perturbation theory running coupling (we use α_s in the $\overline{MS}$-scheme) in the region of its applicability but is well defined for all $Q^2 > 0$.

The main methods of nonperturbative study of the Green's functions in QCD and the running coupling which can be built out of these functions are solving the Dyson–Schwinger (DS) equations and lattice calculations of the functional integrals. In Ref. 24 a summary of the results of such recent studies is given, which can be supplemented with papers [25, 26] where analytic methods were used and [27] with the lattice stimulation results. The variety of the results for the behavior of $\alpha_s(Q^2)$ in the infrared region is connected in particular with different truncation methods applied to close the DS equations. Besides, solving the closed integral equations (or systems of equations) requires, as a rule, the simplifying assumptions quite often breaking the gauge symmetry and the pure technical approximations. It is not surprising, than, that the results of the infrared behavior study of α_s differ greatly from one another and should not be compared literally. Let us note the review papers [28, 29, 30] of the investigations on the IR behavior of the Green's functions, the running coupling in QCD, and their applications in the hadron physics.

The possibility of the singular behavior $\alpha_s \sim 1/Q^2$ at $Q^2 \to 0$ which we consider has been studied in a number of papers. In particular, in Ref. 31 the infrared behavior of the gluon Green's functions was studied by the analytical calculations of the corresponding Feynman integrals of the DS equation for the gluon propagator in the ghost-free axial gauge where the running coupling was defined by the full gluon propagator. It was shown that the singular behavior of the gluon propagator of the form $D(Q) \sim 1/(Q^2)^2$ at $Q^2 \to 0$ was possible but it is essential to give up the commonly used approximation of the three-gluon vertex function by its longitudinal part and to take into account the transverse part of the three-gluon vertex function of a definite form.

This paper is organized as follows. In Section II the one-loop synthetic running coupling model of QCD for all Q^2 is considered. In Section III we study the analytic running coupling for the standard perturbation theory approximations up to four-loop order and its separation into the perturbative and the nonperturbative components. In Section IV the synthetic running coupling with the nonperturbative contributions suppressed at large Q^2 is build on the basis of the analytic running coupling. Setting the parameters of the synthetic running coupling is made. In the concluding section the main results are summarized and some remarks made.

II. ONE-LOOP SYNTHETIC RUNNING COUPLING OF QCD

Let us consider the following additive modification of the one-loop running coupling of QCD by means of the nonperturbative pole type terms[49]

$$\alpha_{\mathrm{syn}}^{(1)}(Q^2) = \frac{4\pi}{b_0} \left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} + \frac{c\Lambda^2}{Q^2} + \frac{(1-c)\Lambda^2}{Q^2 + m_g^2} \right], \quad (1)$$

where the gluon mass parameter

$$m_g = \frac{\Lambda}{\sqrt{c-1}}. \quad (2)$$

Here Q^2 is the Euclidean momentum squared, constant $b_0 = 11 - 2n_f/3$ (n_f is the number of active quark flavors), Λ is the dimensional parameter of the one-loop model (1), c is the dimensionless parameter of this model (it is convenient to introduce the dimensional parameter $\Lambda_1 = \sqrt{c}\Lambda$), $c \in (1, +\infty)$. The parameter Λ can be fixed, for example, by the normalization condition at large Q^2, whereas the parameter c of the model, as can be seen further, describes the relation between the parameter Λ_{QCD} and the string tension parameter σ of the string models[50]. It stands to reason that for the realistic definition of the parameters Λ_{QCD}, σ and their connection it is necessary to go out of the one-loop approximation.

The first term of Eq. (1) is the solution of the renormalization group equation for the QCD running coupling $\alpha_s(Q^2)$

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s) \quad (3)$$

in the one-loop approximation, $\beta(\alpha_s) \simeq -\beta_0 \alpha_s^2$, $\beta_0 = b_0/4\pi$. Introducing the renormalization invariant parameter Λ (the integration constant of the differential equation) we obtain

$$\alpha_s^{(1)}(Q^2) = \frac{4\pi}{b_0} \frac{1}{\ln(Q^2/\Lambda^2)}, \quad (4)$$

where a nonphysical singularity (the Landau pole) at $Q^2 = \Lambda^2$ is present. The vanishing of expression (4) for $Q^2 \to \infty$ corresponds to the remarkable property

of asymptotic freedom [39] of non-Abelian gauge theories, while the growth of α_s (to some critical value or to infinity) with decreasing Q^2 can be connected with the confinement problem.

The pole terms in Eq. (1) are nonperturbative, $\Lambda^2 \simeq \mu^2 \exp\left\{-4\pi/b_0\alpha_s(\mu^2)\right\}$. The sum of the first two terms in Eq. (1) is an analytic function in the complex Q^2-plane with a cut from 0 to $-\infty$,

$$\alpha_{\text{an}}^{(1)}(Q^2) = \frac{4\pi}{b_0}\left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right]. \quad (5)$$

This function can be presented by the dispersion relation

$$\alpha_{\text{an}}^{(1)}(Q^2) = \int_0^\infty \frac{d\sigma\,\tilde{\rho}^{(1)}(\sigma)}{\sigma + Q^2}, \quad (6)$$

where the function $\tilde{\rho}^{(1)}(\sigma)$ is called the one-loop spectral density

$$\tilde{\rho}^{(1)}(\sigma) = \frac{4\pi}{b_0}\frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2}. \quad (7)$$

The last equation can be obtained by the analytic continuation of α_s into the Minkowski space $Q^2 \to -\sigma - i0$ and calculation of the imaginary part $\tilde{\rho}^{(1)}(\sigma) = \frac{1}{\pi}\Im\alpha_s^{(1)}(-\sigma - i0)$. For real $Q^2 > 0$ function (5) is positive monotone decreasing function with maximum at zero $\alpha_{\text{an}}^{(1)}(0) = 4\pi/b_0$. The second nonperturbative term in Eq. (5) does not contribute to the imaginary part of $\alpha_{\text{an}}^{(1)}(Q^2)$ in going to the Minkowski space, so that $\tilde{\rho}^{(1)}(\sigma) = \frac{1}{\pi}\Im\alpha_{\text{an}}^{(1)}(-\sigma - i0)$. The synthetic running coupling (1) can also be presented in the form of the dispersion relation

$$\alpha_{\text{syn}}^{(1)}(Q^2) = \int_{-0}^\infty \frac{d\sigma\,\tilde{\rho}_{\text{syn}}^{(1)}(\sigma)}{\sigma + Q^2}. \quad (8)$$

The function $\tilde{\rho}_{\text{syn}}^{(1)}(\sigma)$ will be called the one-loop spectral density for the synthetic running coupling. It contains additional terms in the form of the delta functions,

$$\tilde{\rho}_{\text{syn}}^{(1)}(\sigma) = \tilde{\rho}^{(1)}(\sigma) + \frac{4\pi}{b_0}\left[c\Lambda^2\delta(\sigma) + (1-c)\Lambda^2\delta(\sigma - m_g^2)\right]. \quad (9)$$

Introducing two pole terms at $Q^2 = 0$ and $Q^2 = -m_g^2 < 0$ does not change the analyticity domain of the analytic running coupling (5). In Eq. (9) for the spectral density the additional terms in the form of two δ-functions localized at $\sigma = 0$ and $\sigma = m_g^2 > 0$ emerged (in comparison with expression (7) of the perturbative origin).

Let us bring equation (1) for $\alpha_{\text{syn}}^{(1)}(Q^2)$ to the explicit renormalization invariant form. It can be done without solving the differential renormalization group equations. Writing the normalization condition for $\alpha_{\text{syn}}^{(1)}(Q^2)$ we obtain an equation for the required dependence of the parameter Λ^2 on the values $\alpha_{\text{syn}}^{(1)}(\mu^2)$ and μ^2 of the form

$$\alpha_{\text{syn}}^{(1)}(\mu^2) = \frac{4\pi}{b_0}\left[\frac{1}{\ln(\mu^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - \mu^2} + \frac{c\Lambda^2}{\mu^2}\right.$$

$$\left. + \frac{(1-c)\Lambda^2}{\mu^2 + m_g^2}\right]. \quad (10)$$

From dimensional considerations

$$\Lambda^2 = \mu^2 \exp\{-\varphi\left(a(\mu^2)\right)\}, \quad (11)$$

where $a(\mu^2) = (b_0/4\pi)\alpha_{\text{syn}}^{(1)}(\mu^2)$. Then for $\varphi(a)$ we have a transcendental equation

$$a = \frac{1}{\varphi(a)} + \frac{1}{1 - e^{\varphi(a)}} + ce^{-\varphi(a)} - \frac{(c-1)^2}{1 + (c-1)e^{\varphi(a)}}. \quad (12)$$

The function $\varphi(a)$ has the behavior $\varphi(a) \simeq 1/a \to +\infty$ as $a \to +0$ for all values of c. This behavior corresponds to the perturbative region. The behavior of this solution at $a \to +\infty$ is $\varphi(a) \simeq -\ln(a/c) \to -\infty$. The beta function $\beta_{\text{syn}}(\alpha_{\text{syn}})$ for the synthetic running coupling can be found by the equation which is analogous to the equation (3),

$$Q^2\frac{\partial\alpha_{\text{syn}}(Q^2)}{\partial Q^2} = \beta_{\text{syn}}(\alpha_{\text{syn}}). \quad (13)$$

Differentiating the running coupling (1) with the use of equations (11) and (12), we obtain

$$\beta_{\text{syn}}(\alpha_{\text{syn}}) = \frac{4\pi}{b_0}\left\{-a + \frac{1}{\varphi(a)} - \frac{1}{\varphi^2(a)} + \frac{1}{(1 - e^{\varphi(a)})^2}\right.$$

$$\left.\left. - \frac{(c-1)^2}{(1 + (c-1)e^{\varphi(a)})^2}\right\}\right|_{a=b_0\alpha_{\text{syn}}/4\pi}. \quad (14)$$

Therefore, using the behavior of function $\varphi(a)$ as $a \to 0$ and ∞, we find the asymptotic behavior

$$\beta_{\text{syn}}(\alpha_{\text{syn}}) \simeq -\frac{b_0}{4\pi}\alpha_{\text{syn}}^2 + o(\alpha_{\text{syn}}^2), \quad \alpha_{\text{syn}} \to 0, \quad (15)$$

$$\beta_{\text{syn}}(\alpha_{\text{syn}}) \simeq -\alpha_{\text{syn}} - \frac{4\pi}{b_0}c(c-2) + o(1), \quad \alpha_{\text{syn}} \to \infty. \quad (16)$$

Doing the corresponding expansions we make sure that the singularity of the β-function (14) at $\varphi \to 0$ is seeming. We also make sure that for all $\alpha_{\text{syn}} > 0$ the function $\beta_{\text{syn}}(\alpha_{\text{syn}})$ is negatively defined.

Let us write the last three terms of the synthetic running coupling (1), taking into account (2), in the form

$$\alpha_{\text{syn}}^{\text{npt}\,(1)}(Q^2) = \frac{4\pi}{b_0}\left[\frac{\Lambda^2}{\Lambda^2 - Q^2} + \frac{c\Lambda^2}{Q^2} + \frac{(1-c)\Lambda^2}{Q^2 + m_g^2}\right]$$

$$= \frac{4\pi}{b_0}\frac{c\Lambda^6}{Q^2(\Lambda^2 - Q^2)(\Lambda^2 + (c-1)Q^2)}. \quad (17)$$

For $a \to +0$, according to equation (12) the function $\phi(a) \simeq 1/a$, thus for Λ^2 from equation (11) we obtain

$\Lambda^2 \simeq \mu^2 \exp(-1/a)$ at $\alpha_{\text{syn}}(\mu^2) = (4\pi/b_0)a \to +0$ so expression (17) must be considered as a nonperturbative component of the synthetic running coupling. The behavior of the nonperturbative "tail" at large Q^2 is the following:

$$\alpha_{\text{syn}}^{\text{npt}\,(1)}(Q^2) = -\frac{4\pi}{b_0}\left[\frac{c}{c-1}\frac{\Lambda^6}{Q^6}\right] + O\left(\frac{\Lambda^8}{Q^8}\right). \qquad (18)$$

As seen from Eq. (18), the nonperturbative contributions of the synthetic running coupling decreases at large Q^2 substantively faster than that of the analytic running coupling (5).

Hence the one-loop synthetic running coupling of QCD has the following interesting properties:

(i) By the construction, as a function of Q^2, it has an analytic structure corresponding to the causality; that is, it is a holomorphic function in the complex Q^2-plane with a cut along the negative real semiaxis.

(ii) As a function of its value $\alpha_{\text{syn}}(\mu^2)$ at the normalization point μ^2, it has an essential singularity at the origin; the asymptotic expansion of its nonperturbative part in $\alpha_{\text{syn}}(\mu^2)$ for $\alpha_{\text{syn}}(\mu^2) \to +0$ is equal to zero, which ensures conformity to the initial perturbation theory.

(iii) In the ultraviolet region, it coincides with usual result of the perturbation theory (with renormalization invariance taken into account) apart from fast decreasing power terms. The nonperturbative component behaves as $\sim 1/(Q^2)^3$ when $Q^2 \to \infty$.

(iv) In the infrared region the synthetic running coupling does not have nonphysical singularities of the perturbation theory. There is the mass term and the singular at the origin term which can be responsible for the confinement of quarks.

As it will be evident from the subsequent considerations, all these properties are valid for the two-, three- and four-loop synthetic running coupling. It is significant that for the one-loop case we have not only representation (8), but for one thing, the nonperturbative contributions are extracted from the synthetic running coupling in the explicit form

$$\alpha_{\text{syn}}^{(1)}(Q^2) = \alpha^{\text{pt}\,(1)}(Q^2) + \alpha_{\text{syn}}^{\text{npt}\,(1)}(Q^2) \qquad (19)$$

and for another, for the nonperturbative contributions, a simple formula is on hand. For large Q^2 ($Q^2 > \Lambda^2$) this contributions can be represented as a series

$$\alpha_{\text{syn}}^{\text{npt}\,(1)}(Q^2) = -\frac{4\pi}{b_0}\left(\frac{\Lambda^2}{Q^2}\right)^3 \sum_{n=0}^{\infty}\left(1 - (1-c)^{-n-1}\right)\left(\frac{\Lambda^2}{Q^2}\right)^n. \qquad (20)$$

For small Q^2 ($Q^2 < \Lambda^2$) we have the expansion

$$\alpha_{\text{syn}}^{\text{npt}\,(1)}(Q^2) = \frac{4\pi}{b_0}\frac{\Lambda^2}{Q^2} \sum_{n=0}^{\infty}\left(1 - (1-c)^{n+1}\right)\left(\frac{Q^2}{\Lambda^2}\right)^n. \qquad (21)$$

The case $c = 2$ is the particular one from the symmetry considerations,

$$\alpha_{\text{syn}}^{\text{npt}\,(1)}(Q^2) = \frac{4\pi}{b_0}\left[\frac{\Lambda^2}{\Lambda^2 - Q^2} + \frac{2\Lambda^2}{Q^2} - \frac{\Lambda^2}{\Lambda^2 + Q^2}\right]$$

$$= \frac{4\pi}{b_0}\frac{2\Lambda^2}{Q^2}\frac{\Lambda^4}{\Lambda^4 - Q^4}, \qquad (22)$$

for which the nonperturbative component is the odd function of Q^2,

$$\alpha_{\text{syn}}^{\text{npt}\,(1)}(-Q^2) = -\alpha_{\text{syn}}^{\text{npt}\,(1)}(Q^2). \qquad (23)$$

Here, in the ultraviolet expansion as well as in the infrared expansion there are no terms of the even powers of Q^2. In particular, the infrared expansion does not have the Coulomb's mode.

Indicate for completeness two boundary cases for the values $c \in (1, +\infty)$ considered. The first case is $c = 1$, for which

$$\alpha_{\text{syn}}^{\text{npt}\,(1)}(Q^2) = \frac{4\pi}{b_0}\frac{\Lambda^4}{Q^2(\Lambda^2 - Q^2)},$$

and the nonperturbative component decreases at infinity not so fast as in expression (18). The second one is $c = \infty$ for which

$$\alpha_{\text{syn}}^{\text{npt}\,(1)}(Q^2) = \frac{4\pi}{b_0}\frac{\Lambda^6}{Q^4(\Lambda^2 - Q^2)},$$

and the singularity in the infrared region is stronger, $\alpha_{\text{syn}}^{(1)}(Q^2) \sim 1/(Q^2)^2$, $Q^2 \to 0$.

III. MULTI-LOOP ANALYTIC RUNNING COUPLING OF QCD

For the multi-loop case the renormalization group equation (3) for the QCD running coupling $\alpha_s(Q^2)$ is of the form

$$Q^2\frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s)$$

$$= -\beta_0\alpha_s^2 - \beta_1\alpha_s^3 - \beta_2\alpha_s^4 - \beta_3\alpha_s^5 + O(\alpha_s^6). \qquad (24)$$

The coefficients β_0, β_1 do not depend on the renormalization scheme choice, whereas the next coefficients do depend on this choice. For the numerical calculations we use its values within the $\overline{MS}$-scheme.

Let us write down the solution of equation (24) for $\alpha_s(Q^2)$ at $L = \ln(Q^2/\Lambda^2) \to \infty$ in the form of the standard expansion in inverse powers of the logarithms

$$\alpha_s(Q^2) = \frac{1}{\beta_0 L} \left\{ 1 - \frac{\beta_1}{\beta_0^2 L} \ln L \right.$$

$$+ \frac{\beta_1^2}{\beta_0^4 L^2} \left[\ln^2 L - \ln L - 1 + \frac{\beta_0 \beta_2}{\beta_1^2} \right]$$

$$- \frac{\beta_1^3}{\beta_0^6 L^3} \left[\ln^3 L - \frac{5}{2} \ln^2 L - \left(2 - \frac{3\beta_0 \beta_2}{\beta_1^2} \right) \ln L \right.$$

$$\left. \left. + \frac{1}{2} - \frac{\beta_0^2 \beta_3}{2\beta_1^3} \right] + O\left(\frac{1}{L^4} \right) \right\}. \tag{25}$$

The sum of the terms of Eq. (25) up to $1/L^n$ order (n=1,2,3,4) will be referred to further on by the n-loop perturbative component of the running coupling and denoted as $\alpha^{\mathrm{pt}}(Q^2)$. It can be written in the form

$$\alpha^{\mathrm{pt}}(Q^2) = \frac{4\pi}{b_0} a(x), \tag{26}$$

$$a(x) = \frac{1}{\ln x} - b \frac{\ln(\ln x)}{\ln^2 x} + b^2 \left[\frac{\ln^2(\ln x)}{\ln^3 x} - \frac{\ln(\ln x)}{\ln^3 x} + \frac{\kappa}{\ln^3 x} \right]$$

$$- b^3 \left[\frac{\ln^3(\ln x)}{\ln^4 x} - \frac{5}{2} \frac{\ln^2(\ln x)}{\ln^4 x} + (3\kappa + 1) \frac{\ln(\ln x)}{\ln^4 x} + \frac{\bar{\kappa}}{\ln^4 x} \right]. \tag{27}$$

Here $x = Q^2/\Lambda^2$ and the coefficients

$$b = \frac{\beta_1}{\beta_0^2}, \quad \kappa = -1 + \frac{\beta_0 \beta_2}{\beta_1^2}, \quad \bar{\kappa} = \frac{1}{2} - \frac{\beta_0^2 \beta_3}{2\beta_1^3}. \tag{28}$$

The values of b_0, b, κ, $\bar{\kappa}$ depend on n_f. Within the standard picture of matching the solutions at heavy quark thresholds the parameter Λ becomes dependent on n_f[51].

The analytic running coupling $\alpha_{\mathrm{an}}(Q^2) = (4\pi/b_0) a_{\mathrm{an}}(x)$ is defined through the dispersion relation

$$a_{\mathrm{an}}(x) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{x + \sigma} \rho(\sigma), \tag{29}$$

where the spectral density $\rho(\sigma) = \Im a_{\mathrm{an}}(-\sigma - i0)$. The analytic approach suggests[52] that $\Im a_{\mathrm{an}}(-\sigma - i0) = \Im a(-\sigma - i0)$. As a result from function $a(x)$ of the form (27) with nonphysical singularities on the positive real semiaxis of the complex plane $x = Q^2/\Lambda^2$ we come to the function $a_{\mathrm{an}}(x)$ of the form (29), which is a single-valued analytic function in the complex plane x with a cut from 0 to $-\infty$ (with a standard definition of the cuts

of the logarithmic function). In Refs. 16, 17 up to four-loop order the separation of the analytic running coupling into the perturbative and nonperturbative components was obtained,

$$\alpha_{\mathrm{an}}(Q^2) = \alpha^{\mathrm{pt}}(Q^2) + \alpha_{\mathrm{an}}^{\mathrm{npt}}(Q^2). \tag{30}$$

In Eq. (30) we take $\alpha^{\mathrm{pt}}(Q^2)$ as the initial (in our case the standard) solution of the renormalization group equation up to four-loop order and $\alpha_{\mathrm{an}}^{\mathrm{npt}}(Q^2)$ appears to arise additionally as a result of the procedure. The following expansion was obtained in the power series

$$\alpha_{\mathrm{an}}^{\mathrm{npt}}(Q^2) = \frac{4\pi}{b_0} \sum_{n=1}^\infty c_n \left(\frac{\Lambda^2}{Q^2} \right)^n, \tag{31}$$

where the coefficients c_n were defined by the beta function coefficients. Note the important properties of expansion (31) such as the higher loop stability of the coefficient of the leading term and the slow increase of the coefficients c_n with number n.

IV. MULTI-LOOP SYNTHETIC RUNNING COUPLING OF QCD

The one-loop synthetic running coupling can be naturally extended to the multi-loop cases. Thus modify the analytic running coupling introducing two additional nonperturbative terms, the singular at zero term of the form $\sim 1/Q^2$ and the mass term of the form $\sim 1/(Q^2 + m_g^2)$. As a result, we come to the expression

$$\alpha_{\mathrm{syn}}(Q^2) = \alpha_{\mathrm{an}}(Q^2) + \frac{4\pi}{b_0} \left[\frac{c\Lambda^2}{Q^2} - \frac{d\Lambda^2}{Q^2 + m_g^2} \right], \tag{32}$$

containing, besides Λ, three subsidiary parameters: c, d and m_g ($m_g \equiv m_\Lambda \Lambda$), which are assumed to be nonzero. We will define this parameters in the following way. At large Q^2 ($Q^2 > \Lambda^2$), using expansion (31), we can find

$$\alpha_{\mathrm{syn}}(Q^2) = \alpha^{\mathrm{pt}}(Q^2) + \frac{4\pi}{b_0} \left[(c_1 + c - d) \frac{\Lambda^2}{Q^2} \right.$$

$$+ (c_2 + dm_\Lambda^2) \left(\frac{\Lambda^2}{Q^2} \right)^2 + (c_3 - dm_\Lambda^4) \left(\frac{\Lambda^2}{Q^2} \right)^3 \right]$$

$$+ O\left(\left(\frac{\Lambda^2}{Q^2} \right)^4 \right). \tag{33}$$

Demand the nonperturbative contributions to be minimal at large Q^2, i.e. the terms of the form $\sim 1/Q^2$, $\sim 1/(Q^2)^2$ to be absent in Eq. (33)[53]. Then two of three parameters are fixed by the following equations

$$d = c + c_1, \quad m_\Lambda^2 = -c_2/(c + c_1). \tag{34}$$

The parameter $\Lambda_1 = \sqrt{c}\Lambda$ will be considered as fixed. The coefficients $c_n < 0$, therefore the absence of tachion condition is $\Lambda < \Lambda_1/\sqrt{-c_1}$. With a given number of loops the free parameter of the synthetic running coupling is only one parameter Λ. Then taking into account Eq. (34) we have

$$\alpha_{\text{syn}}(Q^2) = \alpha^{\text{pt}}(Q^2) + \frac{4\pi}{b_0}\left[c_3 - \frac{c_2^2\Lambda^2}{\Lambda_1^2 + c_1\Lambda^2}\right]\left(\frac{\Lambda^2}{Q^2}\right)^3$$

$$+ O\left((Q^2)^{-4}\right). \qquad (35)$$

As seen from Eq. (35), the leading power nonperturbative term decreases rapidly at $Q^2 \to \infty$ and in the absence of the tachion is negative[54]. For the gluon mass parameter m_g which we call the dynamical gluon mass from Eqs. (34) it follows that

$$m_g = \Lambda\sqrt{\frac{-c_2\Lambda^2}{\Lambda_1^2 + c_1\Lambda^2}}. \qquad (36)$$

Let us turn to the interpretation of the parameter Λ_1 which describes the value of the singular term. As it has already been noted, the behavior of the running coupling $\alpha_V \sim 1/Q^2$ at $Q^2 \to 0$ corresponds to the linear confining quark-antiquark static potential in quenched QCD. We set up a correspondence of the potential and the running coupling using the equation (to compare with Refs. 41 and 36)

$$V(r) = -\frac{4}{3}\int \frac{d^n q}{(2\pi)^n}\exp{(i\mathbf{q}\mathbf{r})}\frac{4\pi\alpha_V(q^2)}{q^2}\bigg|_{n=3}, \qquad (37)$$

where $\alpha_V(q^2)$ is defined as an effective charge [42], which is the renormalization scheme independent and gauge invariant quantity. The color factor corresponds to the $SU(N_c)$ group, $N_c = 3$. Let us assume that in the infrared region

$$\alpha_V(q^2) \simeq \frac{3}{2}\frac{\sigma}{q^2}, \quad q^2 \to 0. \qquad (38)$$

Then the integral over three dimensional momentum space in Eq. (37) formally diverges at the origin. We define this integral introducing the dimensional regularization. After integration over the n-dimensional Euclidean momentum space we put $n = 3$. Then as the divergences do not occur, the transition to the n-dimensional integration provides not only the regularization but the definition of the divergent integral for the case $n = 3$. As a result, for the infrared behavior of the effective charge (38) the behavior of the potential at large distances is as follows:

$$V(r) \simeq \sigma r, \quad r \to \infty, \qquad (39)$$

where $\sigma \equiv a^2$ is the string tension parameter. Let us define the parameter Λ_1 of the synthetic running coupling

α_{syn} from the correspondence of the singular at zero term in Eq. (32) to the infrared behavior (38) of the running coupling α_V. Then

$$\frac{3}{2}\sigma = \frac{4\pi}{b_0}\Lambda_1^2, \quad \Lambda_1^2 = c\Lambda^2. \qquad (40)$$

Therefore, if the string tension parameter is given, the parameter Λ_1 can be specified by Eq. (40). Then with $a \simeq 0.42$ GeV, $b_0 = 9$ we obtain[55] $\Lambda_1 \simeq 435$ MeV. The parameter Λ (as well as the parameter c) can be fixed by the normalization condition, and the synthetic running coupling will be fixed completely.

Consider the dependence of the dynamical gluon mass on Λ for different number of loops of the initial perturbation theory approximation. In Fig. 1 the dynamical gluon mass $m_g(\Lambda)$ is shown for one — four-loop cases. Up to 400 MeV the curves do not diverge too much and at $\Lambda = 375$ MeV $m_g \simeq 0.6$ GeV. We normalize the running cou-

TABLE I: The parameters Λ_{pt} (MeV), Λ_{an} (MeV), Λ_{syn} (MeV), the dynamical gluon mass m_g (MeV) and the parameters c, d on the number of loops. The number of active quark flavors $n_f = 3$, $\Lambda_1 = 435$ MeV. The normalization condition is $\alpha(M_\tau^2) = 0.33$, $M_\tau = 1.777$ GeV.

	1-loop	2-loop	3-loop	4-loop
Λ_{pt}	214.25	395.10	364.19	357.32
Λ_{an}	254.51	636.02	523.86	535.54
Λ_{syn}	214.26	397.10	365.26	358.70
m_g	121.14	648.09	461.66	526.57
c	4.1282	1.2018	1.4204	1.4728
d	3.1282	0.5358	0.7706	0.7447

plings $\alpha_{\text{syn}}(Q^2)$, $\alpha_{\text{an}}(Q^2)$ and $\alpha^{\text{pt}}(Q^2)$ at the τ-lepton mass by [44, 45] $\alpha(M_\tau^2) = 0.33$, $M_\tau = 1.777$ GeV. For this normalization condition the values of the parameters Λ, the dynamical gluon mass m_g and the parameters c, d are given in Table I. Point to two things. The parameters Λ_{syn} and Λ_{pt} are close in value whereas the parameters Λ_{an} are considerably larger. This is a consequence of conditions (34) which give the fast decrease of the nonperturbative terms of $\alpha_{\text{syn}}(Q^2)$ at large Q^2. For all quantities considered the one-loop case turns out to be exceptional, and then the stabilization is observed with the number of loops of the initial perturbation theory approximation.

V. CONCLUSIONS

In the construction of the QCD analytic running coupling (29) the nonphysical singularities of the perturbation theory in the infrared region disappear and in the ultraviolet region the nonperturbative power corrections arise decreasing rapidly at large Q^2, in comparison with the main perturbative component. However, when considering the nonperturbative quantities it may happens
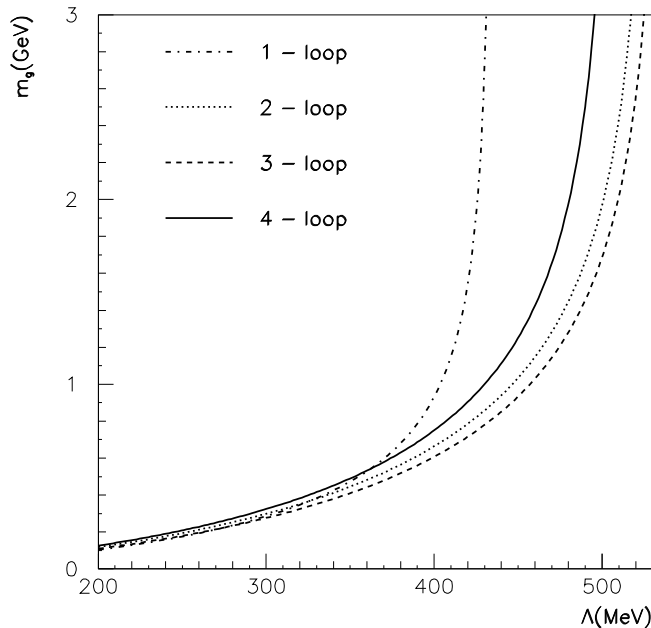
FIG. 1: The gluon mass parameter m_g as a function of Λ for different number of loops of the initial perturbation theory approximation.

that the decrease of the nonperturbative contributions is not fast enough for the consistent definition of these quantities. In the synthetic running coupling it is proposed to provide the highest possible suppression of the nonperturbative contributions at large Q^2 by means of a minimal number of the pole type terms. The parameters characterizing the additional nonperturbative terms have a clear physical meaning and take the reasonable values. The running coupling (1) built on the basis of the analytic running coupling (5) is called the one-loop synthetic running coupling because it contain the parameters related to the ultraviolet region as well as to the infrared region. We introduce the singular at zero term of the form $\sim 1/Q^2$ which can correspond to the linear quark confinement and the mass term of the form $\sim 1/(Q^2 + m_g^2)$ with the parameter m_g corresponding to the non-vanishing dynamical gluon mass. We impose the condition of the fastest decrease of the nonperturbative component at large Q^2 and receive the running coupling model of form (1). The model $\alpha_{\mathrm{syn}}^{(1)}(Q^2)$ has two independent parameters, the dimensional parameter Λ and the dimensionless parameter c which defines the value of the singular term. In Section 2 the one-loop model of the synthetic running coupling and its nonperturbative component properties for $c \in [1, +\infty)$ are considered. A study of the multi-loop analytic running coupling and its nonperturbative component provide a possibility of natural generalization of the synthetic running coupling model

to the multi-loop cases. The multi-loop synthetic running coupling (32) as the one-loop model is constructed by introducing two additional nonperturbative terms of the form $\sim 1/Q^2$ and $\sim 1/(Q^2 + m_g^2)$. The minimality principle of the nonperturbative contributions in the perturbative region leads to two equations (34) for the introduced nonperturbative parameters. As a result, the synthetic running coupling has two independent parameters. First, the parameter Λ which owing to highly fast decrease of the nonperturbative contributions at large Q^2 practically coincides with the parameter Λ_{QCD} in the region of application of the perturbative solutions. Second, the dimensionless parameter c (or the dimensional parameter $\Lambda_1 = \sqrt{c}\Lambda$) determining the value of the singular term. Correlating this parameter responsible for the infrared enhancement by Eqs. (40) with the string tension parameter σ of the string models, we arrive at the dynamical gluon mass as a function of the parameter Λ defined by Eq. (36). The corresponding dependencies for 1–4-loop cases are given in Fig. 1 with $\Lambda_1 = 435$ MeV (that is $\sigma^{1/2} = 0.42$ GeV). The normalization completely defines the synthetic running coupling and for $\alpha(M_\tau^2) = 0.33$ the values of the parameters are shown in Table I. The parameter m_g in Eq. (36) for $\Lambda < \Lambda_1/\sqrt{-c_1}$ is real, and the synthetic running coupling $\alpha_{\mathrm{syn}}(Q^2)$ in Eq. (32) is a holomorphic function in the complex plane Q^2 with a cut along the real negative semiaxis. Thus the synthetic running coupling of QCD $\alpha_{\mathrm{syn}}(Q^2)$ has the properties outlined in Section 2 for the one-loop synthetic running

coupling model.

As seen in Table I the parameters of the synthetic running coupling fixed in such a way show the higher loop stabilization. In particular, the dynamical gluon mass m_g can be estimated as 400–600 MeV. For $\sigma = (0.42$ GeV$)^2$ and $\alpha_{\rm syn}(M_\tau^2) = 0.32, 0.33, 0.34$ it is obtained that $m_g = 453, 527, 613$ MeV (for the 4-loop case) [56]. Hence the string tension identification of the parameter of the synthetic running coupling defining the value of the singular term results in the consistent values for the other parameters considered.

According to Eq. (35) the nonperturbative contributions decrease at large Q^2 as $\sim 1/(Q^2)^3$, which is sufficient for the convergence of the gluon condensate in the ultraviolet region [35]. Thereupon the generalization of the gluon condensate studies [35, 48] to the multi-loop synthetic running coupling is of much interest.

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[49] This one-loop model was considered in Refs. 32, 33, 34 and finally formulated with determination of the parameters from the energy considerations in Ref. 35.

[50] The approximations of QCD which take no account of the effects connected with masses of the heavy quarks contain only one dimensional parameter. For example, it is Λ_{QCD} for large Q^2 or the string tension parameter σ which is adequate for low Q^2. In the potential models the connection of this parameters is studied in description of the bound states of the heavy quarks [36, 37, 38].

[51] In perturbation theory the parameter Λ depends also on the renormalization scheme choice. To study the deviation of the nonperturbative couplings and their consequences from the corresponding perturbative quantities the choice of the standard renormalization scheme $\overline{MS}$ is convenient.

[52] This is precisely the variant of the "analytic improvement" procedure which we consider. The running coupling obtained in this way we call the analytic running coupling.

[53] This condition corresponds to the principle of minimality of the nonperturbative contributions in the perturbative ultraviolet region [33, 34]. Other reasoning is to be used to define the parameters in Eq. (32) if one builds the effective charge for observable with power terms at large Q^2 (see e.g., Ref. 40 for $1/Q^2$ power corrections in $\alpha_{\text{V}}(Q^2)$).

[54] The nonperturbative "tail" as a whole of the running coupling $\alpha_{\text{syn}}(Q^2)$ turns out to be negative at large Q^2.

[55] The string tension parameter $\sigma = (0.42 \text{ GeV})^2$ in particular, in the relativistic string model with massive quarks at the ends of the string [43]. Then the slope of Regge trajectories $\alpha' = 1/(2\pi\sigma) \simeq 0.90 \text{ GeV}^{-2}$.

[56] Estimates of the value of the gluon mass can be found in Ref. 46. Our estimate $m_g = 530 \pm 80$ MeV is in agreement for example with the phenomenological value of the dynamical gluon mass $m_g \approx 400^{+350}_{-100}$ MeV of Ref. 47.